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LINEARIZED SUPERSONIC NONEQUILIBRIUM FLOW PAST AN ARBITRARY BOUNDARY

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ABSTRACT

After a brief review of the inviscid nonequilibrium flow equations in general and of the derivation of the linear nonequilibrium flow equations, the supersonic small-perturbation flow field past an arbitrary boundary is found, by the Laplace transformation, at the vicinity of the initial outgoing frozen Mach line, the vicinity of the boundary, and far downstream of the initial outgoing frozen Mach line. Closed-form solutions are obtained and the general results are applied to compute the geometry of a boundary having constant pressure and the flow at the surface of a biconvex airfoil. A method of extending the results to the entire flow field is indicated.

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SUMMARY

After a brief review of the inviscid nonequilibrium flow equations in general and of the derivation of the linear nonequilibrium flow equations, the supersonic small perturbation flow field past an arbitrary boundary is found, by the Laplace transform method, at the vicinity of the initial outgoing frozen Mach line, the vicinity of the boundary, and far downstream of the initial frozen Mach line.

It is found that all flow quantities at the initial outgoing frozen Mach line decay exponentially with distance above the boundary and that their streamwise derivatives decay more slowly. The flow field far downstream of the initial frozen Mach line is found to be essentially that at equilibrium.

Analytical solutions are obtained for the flow quantities in the vicinity of the boundary for two types of boundary conditions: (a) prescribed boundary geometry and (b) prescribed pressure distribution on the boundary. These general closed form relations are applied to obtain: (a) the geometry of the boundary when the pressure thereon is constant and (b) the flow conditions at the surface of a biconvex airfoil. A method of extending the results to the entire flow field is indicated.

INTRODUCTION

In the classical treatment of gasdynamic problems, it is usually assumed that the flow medium is in thermodynamic and chemical equilibrium. This is a valid assumption only if the relaxation lengths are small. Various authors (refs. 1 through 5) have recently investigated the flow of gases when the flow field is out of equilibrium. The aim of the present

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work is to study the effects of nonequilibrium on some simple flow fields, namely, those where the governing equations can be linearized.

The linearized nonequilibrium flow equation has been derived by, among others, Vincenti (ref. 1), Clarke (ref. 2), and Moore and Gibson (ref. 3). Vincenti solved for the entire flow field over a wavy wall for the complete speed range from subsonic through supersonic. Clarke studied the supersonic flow past a corner; by means of the Laplace transformation, Clarke determined the flow on the wall and in the vicinity of the initial outgoing equilibrium Mach line.

In the present work a somewhat more general type of linearized non- A
equilibrium flow field is studied. The boundary condition is generalized 5
as compared with that of Clarke by allowing the boundary to be arbitrary 1
to a certain extent. Both the direct problem (boundary geometry prescribed) 5
and the inverse problem (flow conditions on the boundary prescribed) are treated. The flow quantities and their derivatives in various regions are obtained; the flow conditions in the vicinity of the initial outgoing frozen Mach line, in the vicinity of the wall, and at large distance downstream of the initial outgoing frozen Mach line are thus determined.

The author is indebted to Dr. Max. A. Heaslet of the Ames Research Center and Professor Walter G. Vincenti of Stanford University for their valuable advice, criticism, and encouragement during the course of the research.

PRINCIPAL SYMBOLS

a	$\dfrac{M_e^2 - 1}{M_f^2 - 1}$
a_e	free-stream sonic speed in the equilibrium condition
a_f	free-stream sonic speed in the frozen condition
A	constant of integration (with respect to η; eq. (19))
c	Clarke's function, defined by equation (32)
C	$\dfrac{(a - 1)(a + 3)}{8}$
D	$\dfrac{(a - 1)(3a + 1)}{8}$
f	any flow quantity

g	function defined by equation (35)
h	specific enthalpy
K	$\dfrac{\tau_\infty U_\infty h_{\rho_\infty}}{h_{\rho_\infty} + h_{q_\infty} q_{e_{\rho_\infty}}}$
L	characteristic geometrical length
l	$\dfrac{L}{K}$
M	free-stream Mach number (frozen or equilibrium)
p	pressure
q	nonequilibrium parameter
q_i	nonequilibrium parameter associated with the ith mode (molecular vibrational energies, degrees of dissociation, and degrees of ionization)
s	Laplace transformation variable
S	specific entropy
t	time; also, dummy variable
u,v	velocities in the x,y directions
U_∞	free-stream velocity
x,y	Cartesian coordinates; x parallel to the free stream
x^+,y^+	normalized Cartesian coordinates: $x^+ \equiv \dfrac{x}{K}$, $y^+ \equiv \dfrac{\beta_f y}{K}$
α	$x^+ - ay^+$
β	$\sqrt{M^2 - 1}$
ξ,η	semicharacteristic coordinates: $\xi \equiv x^+ - y^+$, $\eta \equiv y^+$
θ	slope of the boundary
ρ	density of the fluid

σ $\sqrt{\dfrac{a + s}{1 + s}}$

φ perturbation velocity potential

ψ function defined by equation (33)

ω rates of change of q

Subscripts

o quantity at the origin; also, characteristic constants A515

e equilibrium quantity

f frozen quantity

i associated with the ith mode of nonequilibrium; also, coefficients of the ith term in the series of equation (29)

∞ free-stream quantity

Superscripts

' perturbation quantity

\+ normalized quantity in the x^+,y^+ coordinates

~ normalized quantity in the ξ,η coordinates

— Laplace transform quantity

RÉSUMÉ OF THE LINEARIZED EQUATIONS FOR NONEQUILIBRIUM FLOW

The equations for the nonequilibrium flow of gases, both in the complete and in the linearized forms, have been introduced by various authors (refs. 1 through 5). These governing equations and their linearizations will be reviewed briefly in this section.

Consider the flow of a medium with negligible transport properties (nonviscous, nonconducting, nonradiating) and in the absence of field (or body) forces (e.g., gravitational and electromagnetic). The conservations of mass, momentum, and energy yield

$$\text{Mass} \qquad \frac{D\rho}{Dt} + \rho \operatorname{div} \vec{V} = 0 \tag{1a}$$

$$\text{Momentum} \qquad \frac{D\vec{V}}{Dt} + \frac{1}{\rho} \operatorname{grad} p = 0 \tag{1b}$$

$$\text{Energy} \qquad \frac{Dh}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = 0 \tag{1c}$$

where ρ, V, p, and h are the density, velocity, pressure, and enthalpy of the fluid, and D/Dt denotes the Eulerian differential operator $[(\partial/\partial t) + \vec{V} \cdot \operatorname{grad}]$.

These equations are, of course, the same in the equilibrium, frozen, or the in-between nonequilibrium flow since the conservation principles are not directly dependent on the thermodynamic state of the flow media.

To relate h to p and ρ, which is necessary since we have so far one dependent variable more than the number of equations, we introduce the equation of state in the form

$$\text{State} \qquad h = h(p, \rho, q_1, q_2, \ldots, q_n) \tag{1d}$$

where $q_1, q_2, \ldots, q_n$ are the molecular vibrational energies and degrees of dissociations and ionizations of the constituents of the medium.

Sometimes the state equation is expressed by a set of two equations, such as

$$h = h(p, T, q_1, q_2, \ldots, q_n)$$

$$T = T(p, \rho, q_1, q_2, \ldots, q_n)$$

The first one of the set is commonly called the caloric equation of state, and the second one the thermal equation of state. This manner of expressing the thermodynamic state of the medium is a convenient one when certain idealizations, such as the perfect gas and the Lighthill gas[1] assumptions, are introduced which require the intermediate introduction of temperature. The more compact form of equation (1d) will be adopted here.

Now, along with the introduction of one additional equation (1d), n dependent variables $q_1, q_2, \ldots, q_n$ have been introduced. Obviously more equations are required. These are provided by the rate equations

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[1]Lighthill proposed to approximate the state equation for air by an expression of the form $q^2/(1 - q) = (\rho_d/\rho)\exp(-T_d/T)$, where q is the degree of dissociation, and ρ_d and T_d are constants.

Rate $\quad \dfrac{Dq_i}{Dt} = \omega_i(p, \rho, q_1, q_2, \ldots, q_n) \qquad i = 1, 2, \ldots, n$ (1e)

Equations (1a) through (1e), then, are the complete flow equations which, together with the appropriate boundary and initial conditions, determine the flow field.

Let us compare equations (1) with the limiting cases, equilibrium and frozen. In the equilibrium limit we assume there exists an equilibrium value q_{e_i} for each q_i for the local values of p and ρ; then we have $q_{e_i} = q_{e_i}(p,\rho)$, and equation (1d) can be rewritten as

$$\begin{aligned} h &= h[p, \rho, q_{e_1}(p, \rho), q_{e_2}(p, \rho), \ldots, q_{e_n}(p, \rho)] \\ &= h_e(p, \rho) \end{aligned}$$

whereas in the frozen limit (no reaction), $Dq_i/Dt = 0$, or q_i = constant, and equation (1d) can be rewritten as

$$\begin{aligned} h &= h(p, \rho, q_1 = \text{constant}, q_2 = \text{constant}, \ldots, q_n = \text{constant}) \\ &= h_f(p, \rho) \end{aligned}$$

Thus in the limiting cases of frozen and equilibrium, the equation of state introduces no new variable; consequently equations (1a) through (1d) constitute a complete set of flow equations.

In the case of a perfect gas $q_i = q_i(p, \rho)$ = constant; hence a flow field of perfect gas is in both equilibrium and frozen states simultaneously and cannot be in a nonequilibrium state at any time.

Instead of studying equations (1) in their full forms, we study a much simpler problem, one where the entire flow field perturbs only slightly from a reference, undisturbed, uniform flow. For such a case the flow equations can be linearized. The problems studied will be limited to those of steady ($\partial/\partial t = 0$), two-dimensional planar, supersonic ($M_f > 1$) flow. Also, it will be assumed that only one nonequilibrium effect can be of importance at the same time, that is, $n = 1$, or $h = h(p, \rho, q)$.

Let u and v denote the velocities in the Cartesian coordinates x and y, respectively. The undisturbed velocity U_∞ can be chosen to be in the x direction. The linearization of equations (1) can be done by, first, expanding the flow quantities (dependent variables) in an asymptotic series, that is, taking the first two terms,

$$f \sim f_\infty + f', \quad f' \ll f_\infty \tag{2}$$

and, second, substituting equations (2) into equations (1) and solving for the first order perturbation quantities. Here f is any flow quantity, the subscript ∞ denotes undisturbed stream value, and the prime indicates the first-order perturbation value. In the form of equations (2), the dependent variables are

$$\left.\begin{aligned}\vec{V} &= (U_\infty, 0) + (u', v') \\ p &= p_\infty + p' \\ \rho &= \rho_\infty + \rho' \\ h &= h_\infty + h' \\ q &= q_\infty + q'\end{aligned}\right\} \tag{3}$$

The operator D/Dt can be immediately simplified as

$$\frac{D}{Dt} = U_\infty \frac{\partial}{\partial x}$$

We now substitute equations (3) into equations (1), considering the simple ones first. Integrating equation (1c) and noting h' and p' are zero initially gives

$$\rho_\infty h' = p' \tag{4}$$

Equation (1b) in expanded form is

$$\rho_\infty U_\infty \frac{\partial u'}{\partial x} + \frac{\partial p'}{\partial x} = 0 \tag{5a}$$

$$\rho_\infty U_\infty \frac{\partial v'}{\partial x} + \frac{\partial p'}{\partial y} = 0 \tag{5b}$$

Integrating equation (5a) gives, noting u' is also zero initially,

$$p' = \rho_\infty U_\infty u' \tag{6a}$$

while equations (5b) and (6a) give

$$\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = 0 \tag{6b}$$

Hence, from equations (1b) and (1c) it can be concluded that

(a) The perturbation pressure, enthalpy, and x velocity are proportional to each other.

(b) The flow field is irrotational.

These conclusions are the same regardless of whether the flow is in the frozen, equilibrium, or nonequilibrium state, as long as the disturbances are small.

The remaining equations (mass, state, rate) can be made to yield a relation describing the flow field in terms of the velocities. Since this has already been done in reference 1, the derivation of this relation will not be repeated here. Essentially, equations (1a), (1c), and (1d) are combined, eliminating all variables except the perturbation velocities, which can be put in terms of a perturbation velocity potential (since the flow is irrotational). The resulting equation is

$$K(\beta_f^2\varphi_{xx} - \varphi_{yy})_x + \beta_e^2\varphi_{xx} - \varphi_{yy} = 0 \tag{7}$$

where $K = h_{\rho_\infty}\tau_\infty U_\infty/(h_{\rho_\infty} + h_{q_\infty}q_{\rho_\infty})$, a parameter proportional to the relaxation length $(\tau_\infty U_\infty)$ in the undisturbed stream, τ_∞, usually called the relaxation time, is defined by equation (8) later; x and y are the streamwise and normal coordinates; φ is the perturbation velocity potential, defined by the relations $\varphi_x = u'$ and $\varphi_y = v'$; β_f and β_e are defined by the expressions $\beta_f^2 = M_f^2 - 1$ and $\beta_e^2 = M_e^2 - 1$; and M_f and M_e are the undisturbed stream Mach numbers based on the frozen and equilibrium sonic speeds ($M_f = U_\infty/a_f$, $M_e = U_\infty/a_e$), respectively. The sonic speeds are defined as

$$\text{Frozen} \qquad a_f^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_{S,q}$$

$$\text{Equilibrium} \qquad a_e^2 \equiv \left[\left(\frac{\partial p}{\partial \rho}\right)_S\right]_{q=q_e}$$

S being the entropy of the fluid.

The term $\tau_\infty U_\infty$ comes from the linearization of the rate equation (ref. 1, p. 486) which is, to the first order

$$U_\infty \frac{\partial q'}{\partial x} = \frac{q_e' - q'}{\tau_\infty} \tag{8}$$

Note that q_e' is the fictitious value of q' if q' is in equilibrium with the local p' and ρ'.

From equation (8) it is apparent that when $\tau_\infty U_\infty$ is infinite the rate of change of q is zero and q' takes a constant value; this is the frozen case. If $\tau_\infty U_\infty$ is zero, q' must equal q_e' in order for the rate of change of q to be finite, as it is known to be from classical gasdynamics - that is, q' must take the equilibrium value; this is the equilibrium case.

That the relaxation length plays an important role in determining the nonequilibrium effects is obvious and can be verified in observing the term containing K in equation (7). If $K = 0$, equation (7) reduces to the well-known Prandtl-Glauert linear equation, with the Mach number parameter $\beta = \beta_e$. On the other hand, if $K \to \infty$, the first term dominates, or

$$(\beta_f^2 \varphi_{xx} - \varphi_{yy})_x = 0$$

Integrating this equation once gives

$$\beta_f^2 \varphi_{xx} - \varphi_{yy} = A(y)$$

But $A(y)$ must be zero since it must satisfy all values of x including where the uniform stream is not disturbed. Hence, here equation (7) reduces to the classical Prandtl-Glauert equation again, this time with the Mach number parameter $\beta = \beta_f$.

We complete this section of linearized equations by considering the relations of ρ' and q'. The value of ρ' can be related to the velocity by the mass equation (1a), which is, in the expanded form

$$U_\infty \frac{\partial \rho'}{\partial x} + \rho_\infty \left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \tag{9}$$

or

$$\frac{\rho'}{\rho_\infty} = - \frac{u'}{U_\infty} - \frac{1}{U_\infty} \int \frac{\partial v'}{\partial y} \, dx \tag{9a}$$

The value of q' can be obtained from

$$q' = \left(\frac{\partial q}{\partial p}\right)_\infty p' + \left(\frac{\partial q}{\partial \rho}\right)_\infty \rho' + \left(\frac{\partial q}{\partial h}\right)_\infty h' \tag{10}$$

where p', ρ', and h' are in turn related to the velocities by equations (4), (6a), and (9a).

THE TRANSFORMED EQUATION AND ITS SOLUTIONS

Equation (7) can be solved as follows:

(a) Normalize the equation (eq. (7)).

(b) Transform the equation from (a) by coordinate transformation.

(c) Transform the equation from (b) by the Laplace transformation.

(d) Solve the equation from (c) and invert.

It should be noted that steps (a) and (b) are purely for derivational simplicities, and the use of the Laplace transformation is, of course, only one of the various methods one can use to solve a linear hyperbolic partial differential equation.

The problem to be studied can be stated in mathematical terms as follows:

Let the uniform flow region (fig. 1) be on the left of the origin $(x < 0)$. Let $\theta(x)$ denote the slope of the boundary, which is to be at $y = 0$.[2] The tangency condition, which gives $\theta = v'U_\infty$, requires θ to be zero for $x < 0$. For $x \geq 0$, θ is arbitrary but is assumed to be continuous. The corner $(x = 0,\ y = 0)$ can be either sharp $(\theta(0) \neq 0)$ or smooth $(\theta(0) = 0)$. In supersonic flow it would be sufficient to study only the upper plane, or $y \geq 0$. Finally, the boundary condition on the plane $y = 0$ can be one of the following quantities

(a) $\theta(x)$, or $v'(x, 0)$

(b) $u'(x, 0)$, $p'(x, 0)$, or $h'(x, 0)$

(c) $u_y'(x, 0)$

(d) $v_y'(x, 0)$

etc.

When a boundary condition of the type (a) is prescribed, it is usually called the direct problem, whereas when one of the other types is prescribed, it is usually called the inverse problem. In the present work, both the direct problem (type (a) boundary condition prescribed) and an inverse problem (type (b) boundary condition prescribed) are treated.

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[2]Within the accuracy of the linearized theory, the boundary condition can be specified on $y = 0$ rather than on the actual boundary itself.

The boundary conditions can be summarized as

$$\varphi = \begin{cases} 0 & x < 0 \\ \text{finite} & x \to \infty\,, \quad y \to \infty \end{cases}$$

$$\frac{v'}{U_\infty} = \theta(x) \qquad \eta = 0$$

With the introduction of the new variables φ^+, x^+, y^+, and a, defined by the relations

$$\varphi = U_\infty K\varphi^+\,, \qquad x = Kx^+\,, \qquad y = (K/\beta_f)y^+\,, \qquad a = \beta_e{}^2/\beta_f{}^2$$

equation (7) becomes

$$(\varphi^+_{x^+x^+} - \varphi^+_{y^+y^+})_{x^+} + a\varphi^+_{x^+x^+} - \varphi^+_{y^+y^+} = 0 \tag{11}$$

Thus the flow equation can be normalized into an equation containing a single parameter a, although in general the boundary conditions may introduce more parameters, such as a characteristic geometrical length (e.g., the chord length of an airfoil). Note that, since a_f is always greater than a_e (ref. 4, p. 15), a is always greater than unity.

The dimensionless coordinates x^+ and y^+ are now in terms of numbers of relaxation lengths instead of physical lengths. If the characteristic geometrical length is large compared with the relaxation length, the fluid particles at any point (x^+, y^+) in the flow field, except in the neighborhood of the origin, have traveled a distance of a large number of relaxation lengths and hence have essentially arrived at the equilibrium condition; this is formally equivalent to the case where the relaxation length (and hence K) is small. On the other hand, if the characteristic geometrical length is small compared with the relaxation length, the fluid particles in the vicinity[3] of the boundary have traveled a distance small compared with the relaxation length; thus the flow field in this vicinity is essentially at the frozen condition. This is formally equivalent to the case where the relaxation length is large. Thus in the cases where there exists a geometrical characteristic length, the flow field in the region of interest is in the equilibrium, nonequilibrium, or frozen condition depending on whether l ($\equiv L/K$, denoting L the geometrical characteristic length) is ∞, finite, or 0, respectively. On the other hand, in the cases where there is no characteristic geometrical length, all three conditions (equilibrium, nonequilibrium, and frozen) exist;

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[3] Even when the characteristic geometrical length is small, the flow field at large y behaves like that at equilibrium, and in some cases (e.g., an airfoil where θ = constant for x greater than the chord length) the flow field behaves like that at equilibrium at large x.

the frozen condition exists near the corner ($x^+ = 0$, $y^+ = 0$), the non-equilibrium condition exists at finite positive x^+ and y^+, and the equilibrium condition occurs at $x^+ \to \infty$ and $y^+ \to \infty$.

Now transform equation (11) by the change of independent variables $\xi = x^+ - y^+$, $\eta = y^+$. This is not an absolutely necessary step, but it simplifies the analysis. It is easy to see that the disturbances (from the wall) cannot affect the flow field ahead of the frozen Mach line originating at the corner ($x^+ = 0$, $y^+ = 0$); that is, the disturbances (φ^+, u^+, v^+, and their derivatives) are zero in front of the first frozen Mach line ($x^+ - y^+ = 0$). It is convenient, therefore, to refer to a location in the flow field by its distance ($x^+ - y^+$) downstream from the initial frozen Mach line, and its distance (y^+) above the wall. Other coordinate systems with a different choice for the second coordinate, such as $x^+ - y^+$ and $x^+ + y^+$, can be also used; but for the types of problems we are considering, single wall with outgoing wave (of disturbances), this choice of $\xi = x^+ - y^+$ and $\eta = y^+$ is simpler.

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Applying the foregoing coordinate transformation, noting $\partial/\partial x^+ = \partial/\partial\xi$ and $\partial/\partial y^+ = \partial/\partial\eta - \partial/\partial\xi$, and denoting $\tilde{f}(\xi,\eta) = f^+(x^+, y^+)$ converts equation (11) to

$$2\tilde{\varphi}_{\eta\xi\xi} - \tilde{\varphi}_{\eta\eta\xi} + (a - 1)\,\tilde{\varphi}_{\xi\xi} + 2\tilde{\varphi}_{\eta\xi} - \tilde{\varphi}_{\eta\eta} = 0 \tag{12}$$

The boundary conditions are now

$$\tilde{\varphi} = \begin{cases} 0 & \xi < 0 \quad (13a) \\ \text{finite} & \eta \to \infty \quad (13b) \end{cases}$$

and

$$\tilde{\varphi}_{\eta} - \tilde{\varphi}_{\xi} = \tilde{\theta}(\xi) \qquad \eta = 0 \tag{14}$$

Equation (14) is the flow tangency condition at the wall. Again, remember that it need not be always that the boundary slope, $\tilde{\theta}$, is prescribed. In the inverse problem, some other flow quantity ($\tilde{u}$, $\tilde{v}_{\eta}$, etc.) on the boundary can be prescribed and the rest of the flow quantities (including $\tilde{\theta}$) are to be determined. Nevertheless, the tangency condition (eq. (14)) still holds regardless of whether $\tilde{\theta}$ is a known or an unknown function in a problem. For later interpretation of the results, it is necessary to know the normalized quantities, $\tilde{f}(\xi,\eta)$, in terms of the physical quantities, $f(x,y)$; they are related as follows:

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$$\left.\begin{aligned}
\tilde{\varphi} &= \varphi/U_\infty K \\
\tilde{u} &= (u - U_\infty)/U_\infty \\
\tilde{v} &= v/U_\infty \\
\tilde{\theta} &= \theta/\beta_f \\
\tilde{p} &\equiv p'/\rho_\infty U_\infty^2 = (p - p_\infty)/\rho_\infty U_\infty^2 \\
\tilde{\rho} &\equiv \rho'/\rho_\infty = (\rho - \rho_\infty)/\rho_\infty \\
\tilde{q} &\equiv q'/q_\infty = (q - q_\infty)/q_\infty
\end{aligned}\right\} \tag{15}$$

We now solve equation (12) by means of the Laplace transformation (with respect to ξ). This is the same general approach used by Clarke in solving the flow past a straight corner. The present work considers a somewhat more general type of boundary condition since the shape of the wall can be arbitrary, so long as it is smooth. By means of the Laplace transformation, the partial differential equation (12) is transformed into an ordinary differential equation, the solution of which gives the results in terms of the transformed variables. The inversion of these transformed solutions gives the desired results in physical terms. As will be seen later, the last step is the most difficult.

The Laplace transform of a function $\tilde{f}(\xi,\eta)$ with respect to ξ is defined as

$$L[\tilde{f}(\xi,\eta)] \equiv \bar{f}(s,\eta) \equiv \int_0^\infty \exp(-s\xi)\tilde{f}(\xi,\eta)d\xi$$

where L and the bar denote the Laplace transform.

Only a few identities and theorems of the Laplace transformation will be used in this paper. For later reference they are listed in the following:

$$L\left[\frac{\partial^n}{\partial\xi^n}\tilde{f}(\xi,\eta)\right] = s^n\bar{f}(s,\eta) - \sum_{j=0}^{n-1} s^{n-1-j}\frac{\partial^j}{\partial\xi^j}\tilde{f}(0,\eta) \tag{16a}$$

$$L\left[\frac{\partial^n}{\partial\eta^n}\tilde{f}(\xi,\eta)\right] = \frac{\partial^n}{\partial\eta^n}\bar{f}(s,\eta) \tag{16b}$$

$$L^{-1}[s\bar{f}_1(s)\bar{f}_2(s)] = \tilde{f}_1(0)\tilde{f}_2(\xi) + \int_0^\xi \tilde{f}_1'(t)\tilde{f}_2(\xi - t)dt \tag{16c}$$

$$\tilde{f}(0,\eta) = \lim_{s \to \infty} s\bar{f}(s,\eta) \tag{16d}$$

$$\lim_{\xi \to \infty} \tilde{f}(\xi,\eta) = \lim_{s \to 0} s\bar{f}(s,\eta) \tag{16e}$$

Applying the Laplace transformation to equation (12), using the identities (16a) and (16b) and part of the boundary condition equation (13), we obtain

$$\bar{\varphi}_{\eta\eta} - 2s\bar{\varphi}_{\eta} - \left(\frac{a - 1}{1 + s}\right) s^2\bar{\varphi} = -(1 + s)^{-1} \left\{\left[2 \frac{\partial}{\partial\eta} + (a - 1)\right] \tilde{\varphi}_{\xi}(0,\eta) - \left[\frac{\partial^2}{\partial\eta^2} - 2(1 + s) \frac{\partial}{\partial\eta} + (a - 1)s\right] \tilde{\varphi}(0,\eta)\right\} \tag{17}$$

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Setting the right-hand side of equation (17) zero gives

$$\bar{\varphi}_{\eta\eta} - 2s\bar{\varphi}_{\eta} - [(a - 1)/(1 + s)]s^2\bar{\varphi} = 0 \tag{18}$$

This last assumption is valid if it can be shown that equation (18) yields a solution such that when it is substituted into the right-hand side of equation (17) the latter expression vanishes. Later, we shall see that solution of equation (18) gives (eqs. (24) and (25))

$$\tilde{\varphi}(0,\eta) = 0$$

$$\tilde{\varphi}_{\xi}(0,\eta) = \tilde{\varphi}_{\xi}(0,0)\exp\left(- \frac{a - 1}{2} \eta\right)$$

Hence, the right-hand side of equation (17) does indeed vanish.

The general solution of equation (18) is of the form (denoting $\sigma \equiv [(a + s)/(1 + s)]^{1/2}$)

$$\bar{\varphi} = A(s)\exp[-s\eta(\sigma - 1)] + B(s)\exp[s\eta(\sigma - 1)] \tag{19}$$

where A and B are functions of s to be determined from the remaining boundary conditions.

Since a is always greater than unity, $\sigma - 1$ is always positive. On the upper plane ($\eta \geq 0$), the second term grows unbound with η unless B is zero. Since $\tilde{\varphi}$ (hence $\bar{\varphi}$) is finite at $\eta \to \infty$, B(s) must be zero. The wall tangency condition (eq. (14)), in transformed terms, is

$$\overline{\varphi}_\eta - s\overline{\varphi} + \tilde{\varphi}(0,\eta) = \overline{\theta} \quad \text{at} \quad \eta = 0$$

This wall tangency condition determines $A(s)$ in terms of $\overline{\theta}(s)$, resulting in an expression for $\overline{\varphi}$ in terms of $\overline{\theta}(s)$; we thus obtain finally

$$\overline{\varphi}(s,\eta) = -s^{-1}\overline{\theta}\sigma^{-1}\exp[-s\eta(\sigma - 1)] \tag{20}$$

This is the general solution, in terms of Laplace transforms, of the problem stated at the beginning of this section. The inversion of this expression yields the solution in the physical terms. Unfortunately, the inversion of equation (20) is rather difficult in general.

For $s \to 0$, $s \to \infty$, or $\eta \to 0$, the right-hand side of equation (20) can be simplified considerably, resulting in expressions that can be inverted either directly, with the aid of equations (16c), (16d), and (16e), or with the aid of inversion tables (ref. 6). In the physical plane, $s \to 0$ corresponds to $\xi \to \infty$, and $s \to \infty$ corresponds to $\xi \to 0$. Thus, we can readily obtain the solution in physical terms in the vicinity of the wall ($\eta \to 0$), in the vicinity of the initial frozen Mach line ($\xi \to 0$) and at large distances downstream of the initial frozen Mach line ($\xi \to \infty$). This will be shown in detail in the following sections.

VICINITY OF THE INITIAL FROZEN MACH LINE

To determine the flow field in the vicinity of the initial frozen Mach line we compute $\tilde{\varphi}$, $\tilde{u}$, $\tilde{v}$ and their ξ derivatives (first order or higher) at $\xi = 0$. This can be done by expanding the expressions for the Laplace transforms of $\tilde{\varphi}$, $\tilde{u} = \varphi^+_{x^+} = \tilde{\varphi}_\xi$, $\tilde{v} = \varphi^+_{y^+} = \tilde{\varphi}_\eta - \tilde{\varphi}_\xi$, and their ξ derivatives individually for large s, then applying the theorem of equation (16d). It will be sufficient to show the actual derivations for $\tilde{\varphi}(0,\eta)$, $\tilde{u}(0,\eta)$, and $\tilde{u}_\xi(0,\eta)$ since the higher ξ derivatives of $\tilde{u}$ and the similar expressions for $\tilde{v}$ are obtained in exactly the same manner.

In the vicinity of the corner ($\xi \to 0$) the boundary slope can be expanded into the form

$$\tilde{\theta} \sim \begin{cases} \tilde{\theta}_o + \tilde{\theta}_o'\xi + \frac{1}{2}\tilde{\theta}''\xi^2 + \ldots & \xi \geq 0 \\ 0 & \xi < 0 \end{cases}$$

or

$$\overline{\theta} \sim \tilde{\theta}_o s^{-1} + \tilde{\theta}_o' s^{-2} + \tilde{\theta}_o'' s^{-3} + \ldots \tag{21}$$

At large s, $\sigma(s)$ and $\sigma^{-1}(s)$ can be expanded as

$$\sigma \sim 1 + \frac{a-1}{2} s^{-1} - Ds^{-2} + O(s^{-3}) \tag{22}$$

$$\sigma^{-1} \sim 1 - \frac{a-1}{2} s^{-1} + Es^{-2} + O(s^{-3}) \tag{23}$$

where $D \equiv (a-1)(a+3)/8$, $E \equiv (a-1)(3a+1)/8$.

Substituting equations (21), (22), and (23) for $\overline{\theta}$, σ, and σ^{-1} into equation (20) and applying the theorem of equation (16e) gives

$$\tilde{\varphi}(0,\eta) = 0 \tag{24}$$

Substituting these same expanded quantities into $\overline{u} = s\overline{\varphi} - \tilde{\varphi}(0,\eta)$ and applying again the theorem of equation (16d) gives

$$\tilde{u}(0,\eta) = \tilde{\theta}_o \exp\left(-\frac{a-1}{2}\eta\right) \tag{25}$$

The same operation (asymptotic inversion) on $(\overline{u}_\xi) = s^2\overline{\varphi} - s\tilde{\varphi}(0,\eta) - \tilde{u}(0,\eta)$ yields

$$\tilde{u}_\xi(0,\eta) = -\left(\tilde{\theta}_o' - \frac{a-1}{2}\tilde{\theta}_o + D\tilde{\theta}_o\eta\right)\exp\left(-\frac{a-1}{2}\eta\right) \tag{26}$$

Equations (24) and (25) confirm the assumptions needed to obtain equation (18) of the previous section.

One can in the same manner invert the Laplace transform of higher ξ derivatives of $\tilde{u}$ and the ξ derivatives of $\tilde{v}$. In summary we obtain

$$\left.\begin{matrix} -\tilde{u}(0,\eta) \\ \overline{v}(0,\eta) \end{matrix}\right\} = \tilde{\theta}_o \exp\left(-\frac{a-1}{2}\eta\right) \tag{27}$$

$$\left.\begin{matrix} -\tilde{u}_\xi(0,\eta) \\ \tilde{v}_\xi(0,\eta) \end{matrix}\right\} = \left\{\begin{matrix} \left(\theta_o' - \frac{a-1}{2}\theta_o + D\theta_o\eta\right) \\ (\theta_o' + D\theta_o\eta) \end{matrix}\right\} \exp\left(-\frac{a-1}{2}\eta\right) \tag{28}$$

or, in general,

$$\left[\frac{\partial^n}{\partial\xi^n}(\tilde{u} \text{ or } \tilde{v})\right]_{\xi=0} = \sum_{i=0}^{n} c_i \eta^i \exp\left(-\frac{a-1}{2}\eta\right) \tag{29}$$

where the c_i are constants depending on i. Thus $\tilde{u}$ and $\tilde{v}$ and their ξ derivatives (to all orders) vanish at large η in the vicinity of the initial frozen Mach line. This is expected since at large η the flow field is essentially at the equilibrium condition, and all disturbances should approach zero in front of the neighborhood of the initial equilibrium Mach line, which is at $\xi = (\sqrt{a} - 1)\eta$.

The flow field in the vicinity of the frozen Mach line can be described in general by the expression ($f = \tilde{u}$ or $\tilde{v}$)

$$f(\xi,\eta) = f(0,\eta) + f_\xi(0,\eta)\xi + \ldots \tag{30}$$

where f, f_ξ, and so forth are obtained in the manner shown above.

It should be noted that neither the classical (equilibrium or frozen) nor the present nonequilibrium linear theory is accurate at large distances from the wall because, owing to cumulative second order effects, the linear theory is not uniformly correct at large η. The present section, however, shows the essential effects of nonequilibrium in the linear theory, without taking into account the nonlinearity. In order to obtain results that are uniformly valid for all η, nonlinear effects would have to be taken into account at least approximately.

FLOW FIELD IN THE VICINITY OF THE WALL

To determine the flow field in the vicinity of the wall, we compute the flow quantities and their η derivatives at $\eta = 0$. Then, letting $\tilde{f}$ denote any flow quantity in the vicinity of the wall, $\tilde{f}$ is given by the series $\tilde{f}(\xi,\eta) = \tilde{f}(\xi,0) + \tilde{f}_\eta(\xi,0) + \ldots$. The quantities $f(\xi,0)$, $\tilde{f}_\eta(\xi,0)$, . . . can be obtained by inverting $\bar{f}(s,0)$, $\bar{f}_\eta(s,0)$, . . ., etc.

Consider $\tilde{u}$, $\tilde{v}$, their first ξ derivatives, and $\tilde{\theta}$, as examples. The Laplace transforms of these quantities at $\eta = 0$ are given by the relations deduced from equation (20) as:

$$\bar{u}(s,0) = -\bar{\theta}\sigma^{-1} \tag{31a}$$

$$\bar{v}(s,0) = \bar{\theta} \tag{31b}$$

$$\bar{u}_\eta(s,0) = s\bar{\theta}(1 - \sigma^{-1}) \tag{31c}$$

$$\bar{v}_\eta(s,0) = -s\bar{\theta}(\sigma - 1) \tag{31d}$$

For the direct problem we can invert equations (31) in their present forms and obtain $\tilde{u}$, $\tilde{v}$, etc., in terms of $\tilde{\theta}$, and for the inverse problem we can rewrite equation (31a) with $\tilde{\theta}$ in terms of $\tilde{u}$, then invert and obtain $\tilde{\theta}$ in terms of $\tilde{u}(\xi,0)$.

The inversion of σ and σ^{-1} can be obtained from an inverse transform table (e.g., ref. 6), and the inverse transform of $\bar{\theta}$ is simply $\tilde{\theta}$ by definition. The expressions of equations (31), however, contain products of two functions of s (except eq. (31b)). To invert this type of expression, one can apply equation (16c), the theorem for inversion of products.

In the direct problem for $\tilde{u}$, let $\tilde{f}_1 = \tilde{\theta}$ or $\bar{f}_1 = \bar{\theta}$, then $\bar{f}_2$ is identified as $s^{-1}\sigma^{-1}$. The quantity $\tilde{f}_2$ is obtained by inverting $s^{-1}\sigma^{-1}$. The inversion of $s^{-1}\sigma^{-1}$ and $s^{-1}\sigma$ (for later use) can be done in the following manner. From reference 6, page 235, we obtain

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$$L^{-1}\left[s\left(\frac{b+s}{c+s}\right)^{1/2}\right] = \exp\left(-\frac{c+b}{2}\xi\right)I_0\left(\frac{c-b}{2}\xi\right) + b\int_0^{\xi}\exp\left(-\frac{c+b}{2}\omega\right)I_0\left(\frac{c-b}{2}\omega\right)d\omega$$

where I_0 is the zeroth-order modified Bessel function of the first kind. The above relation can be used to evaluate the inverse of $s^{-1}\sigma^{-1}$ and $s^{-1}\sigma$

$$c(\xi;a) \equiv L^{-1}(s^{-1}\sigma^{-1}) = \exp\left(-\frac{a+1}{2}\xi\right)I_0\left(\frac{a-1}{2}\xi\right) + \int_0^{\xi}\exp\left(-\frac{a+1}{2}\omega\right)I_0\left(\frac{a-1}{2}\omega\right)d\omega \tag{32}$$

$$\psi(\xi;a) \equiv L^{-1}(s^{-1}\sigma) = \exp\left(-\frac{1+a}{2}\xi\right)I_0\left(\frac{1-a}{2}\xi\right) + a\int_0^{\xi}\exp\left(-\frac{1+a}{2}\omega\right)I_0\left(\frac{1-a}{2}\omega\right)d\omega \tag{33}$$

One can easily verify that

$$\psi(\xi;a) \equiv c(a\xi;a^{-1})$$

Thus $\tilde{u}$ in terms of $\tilde{\theta}$ is

$$\tilde{u}(\xi,0) = -\tilde{\theta}(0)c(\xi) - \int_0^{\xi}\tilde{\theta}'(t)c(\xi-t)dt \tag{34a}$$

Equation (31b) gives immediately

$$\tilde{v}(\xi,0) = \tilde{\theta}(\xi) \tag{34b}$$

For $\tilde{u}_\eta(\xi,0)$, note that, from equation (31c), $\bar{u}_\eta(s,0) = s\bar{\theta} - s\bar{\theta}\sigma^{-1}$; the inverse transform of the first term gives $\tilde{\theta}'$, and the inverse transform of the second term gives $\tilde{u}_\xi(\xi,0)$. Thus $\tilde{u}_\eta(\xi,0)$ is

$$\tilde{u}_\eta(\xi,0) = \tilde{\theta}'(\xi) + \tilde{u}_\xi(\xi,0) \tag{34c}$$

For $\tilde{v}_\eta(\xi,0)$, note that (eq. (31d)) $\tilde{v}_\eta(s,0) = s\bar{\theta} - s\bar{\theta}\sigma$. Again, the inverse transform of the first term gives $\tilde{\theta}'(\xi)$, and the inverse transform of the second term gives $g'(\xi)$, where $g(\xi)$ is defined as

$$g(\xi) \equiv \tilde{\theta}(0)\psi(\xi) - \int_0^\xi \tilde{\theta}'(t)\psi(\xi - t)dt \tag{35}$$

Thus $\tilde{v}_\eta(\xi,0)$ is

$$\tilde{v}_\eta(\xi,0) = \tilde{\theta}'(\xi) - g'(\xi) \tag{34d}$$

Note that $\tilde{\rho} = -\tilde{u} - \beta_f{}^2\int_0^\xi (\tilde{v}_\eta - \tilde{v}_\xi)d\xi$, the density is therefore

$$\tilde{\rho}(\xi,0) = -\tilde{u}(\xi,0) + \beta_f{}^2 g(\xi) \tag{34e}$$

For the inverse problem we rewrite equation (31a) as

$$\bar{\theta}(s) = -\bar{u}(s,0)\sigma(s)$$

and apply equation (16d), taking $\tilde{f}_1$ to be $\tilde{u}(\xi,0)$ or $\bar{f}_1 = \bar{u}(s,0)$ (in which case $\bar{f}_2$ is identified as $s^{-1}\sigma$), and obtain

$$\tilde{\theta}(\xi) = -\tilde{u}(0,0)\psi(\xi) - \int_0^\xi \tilde{u}_t(t,0)\psi(\xi - t)dt \tag{36}$$

It is interesting to note that, while in the classical linear theory $\tilde{u}(\xi,0) \sim \tilde{v}(\xi,0)$, in the nonequilibrium linear theory the same result does not hold; that is, the reciprocal theorem in classical linear theory,

Frozen $\qquad \tilde{u}(\xi,0) - \tilde{v}(\xi,0) = 0$

Equilibrium $\qquad \sqrt{a}\,\tilde{u}(\xi,0) - \tilde{v}(\xi,0) = 0$

is no longer valid. On the other hand, noting that I_0 is an even function, one can see that

$$c - \psi = (1 - a) \int_0^{\xi} \exp\left(- \frac{a + 1}{2} \omega\right) I_0 \left(\frac{a - 1}{2} \omega\right) d\omega$$

Hence $c = \psi$ when $a = 1$, and, from a comparison of equations (34) and (35), it is apparent that when $a = 1$, $\tilde{u}(\xi,0) - \tilde{v}(\xi,0) = 0$. Physically, $a = 1$ when the two sonic speeds (frozen and equilibrium) approach the same value. In this case the frozen flow value is the same as the equilibrium flow value; hence when $a = 1$ the flow is always in both frozen and equilibrium conditions, and there is no nonequilibrium effect.[4]

Examples

Equations (34) give the distributions of various flow quantities on an arbitrary smooth boundary, and equation (36) gives the shape of the boundary for an arbitrary distribution of $\tilde{u}(\xi,0)$. We now apply these relations to a few specific problems.

Flow past a straight corner.- This case was treated by Clarke in reference 2. For a straight corner, $\tilde{\theta}$ = constant. Letting $\tilde{\theta}_o$ denote this constant, equations (34) give

$$\begin{aligned}
\tilde{u}(\xi,0) &= -\tilde{\theta}_o c(\xi) \\
\tilde{v}(\xi,0) &= \tilde{\theta}_o \\
\tilde{u}_\eta(\xi,0) &= -\tilde{\theta}_o c'(\xi) \\
\tilde{v}_\eta(\xi,0) &= -\tilde{\theta}_o \psi'(\xi)
\end{aligned}$$

Thus $c(\xi)$ can be interpreted as the response of $\tilde{u}(\xi,0)$ per unit negative constant deflection angle. From these expressions the thermodynamic properties can also be obtained. For examples, the pressure and density are

$$\frac{p - p_\infty}{\rho_\infty U_\infty{}^2} = \frac{h - h_\infty}{u_\infty{}^2} = -\tilde{u} = \frac{\theta_o}{\beta_f} c\left(\frac{x}{K}\right)$$

$$\frac{\rho - \rho_\infty}{\rho_\infty} = -\tilde{u} + \beta_f{}^2 g = \frac{\theta_o}{\beta_f}\left[c\left(\frac{x}{K}\right) + \beta_f{}^2\psi\left(\frac{x}{K}\right)\right]$$

The discussions on these results will not be elaborated since they were already treated in reference 2. The functions $c(\xi)$ and $\psi(\xi)$ are not common table functions. For the present paper they are computed by

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[4]This can also be verified from the original differential equation. Integrating equation (11) once gives, for $a = 1$,

$$\varphi^+_{x^+x^+} - \varphi^+_{y^+y^+} = f(y^+)\exp(-x^+)$$

since $f(y^+)$ must satisfy all values of x^+, and $\varphi^+_{x^+x^+}$ and $\varphi^+_{y^+y^+}$ are zero for $x^+ < 0$, $f(y^+)$ must be zero.

means of the IBM 704 digital computer of the Ames Research Center. These functions are tabulated in tables I and II and plotted in figure 2 versus x for $a = 2$ and a range of values of K.

As one can see from the tables, c approaches $\sqrt{1/a}$ while ψ approaches $\sqrt{a}$ as ξ approaches ∞. With some simple computations it is easy to verify that $(p - p_\infty)/\rho_\infty U_\infty^2$ and $(\rho - \rho_\infty)/\rho_\infty$ approach the equilibrium values at $\xi \to \infty$, which confirm our earlier physical deduction. More considerations on the flow field at $\xi \to \infty$ will be treated in the next section.

Wall with a constant pressure.- In classical gasdynamics, a constant pressure on the boundary past the corner is associated with a boundary of constant deflection angle. This is not the case when the flow is out of equilibrium.

Letting $\tilde{p}_o$ denote the constant pressure, equation (36) gives (noting $\tilde{p} = -\tilde{u}$)

$$\tilde{\theta}(\xi) = \tilde{p}_o\psi(\xi)$$

or, in terms of physical values,

$$\frac{\theta}{\beta_f} = \frac{p - p_\infty}{\rho_\infty U_\infty^2}\,\psi\,\frac{x}{K} \tag{37}$$

The integration of θ from equation (37) gives the shape of the boundary as

$$y = K\beta_f\,\frac{p - p_\infty}{\rho_\infty U_\infty^2}\int_0^{x/K}\psi(t)dt \tag{38}$$

The value of $y/[K\beta_f(p - p_\infty)/\rho_\infty U_\infty^2]$ has been computed by integrating $\psi(\xi)$ numerically, and the result is presented in figure 3. At the vicinity of the corner, that is, for small x/K, the fluid has little chance to adjust its flow properties to conform with the new environment (pressure rise); consequently, the boundary starts out following essentially the boundary of the frozen case. As the fluid flows downstream, its flow properties tend toward equilibrium. Thus the initial boundary slope takes the frozen value, and the boundary becomes parallel to the equilibrium case far downstream.

Flow past a class of walls with varying slope.- Consider now, as the last example of this section, a class of boundaries whose shapes are defined by the relation

$$\tilde{\theta} = \tilde{\theta}_o[1 - (n + 1)(\xi/l)^n]$$

or

$$\tilde{\theta}' = -\tilde{\theta}_o\,\frac{(n + 1)n}{l}\left(\frac{\xi}{l}\right)^{n-1}$$

From equation (34a) we obtain the velocity $\tilde{u}$ on the wall as

$$\tilde{u}(\xi,0) = -\tilde{\theta}_0 c(\xi) + [\tilde{\theta}_0(n + 1)n/l] \int_0^{\xi} (t/l)^{n-1} c(\xi - t)dt \tag{39}$$

The pressure on the boundaries is, since $\tilde{p} = -\tilde{u}$

$$\frac{(p - p_\infty)\beta_f}{\rho_\infty U_\infty^2 \theta_0} = c\left(\frac{x}{K}\right) - \frac{(n + 1)n}{L^n} \int_0^{x/K} t^{n-1}\, c\left(\frac{x}{K} - t\right) dt \tag{40}$$

The density on the wall is, noting

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$$g(\xi) = \tilde{\theta}_0 \psi(\xi) + [\tilde{\theta}_0(n + 1)n/l] \int_0^{\xi} (t/l)^{n-1} \psi(\xi - t)dt$$

and $\tilde{\rho}(\xi,0) = -\tilde{u}(\xi,0) + \beta_f^2 g(\xi)$ (eq. (34e))

$$\frac{\rho - \rho_\infty}{\rho_\infty} \frac{\beta_f}{\theta_0} = c\left(\frac{x}{K}\right) + \beta_f^2 \psi\left(\frac{x}{K}\right) - \frac{n + 1}{l^n}\, n \int_0^{x/K} t^{n-1}\left[c\left(\frac{x}{K} - t\right) + \beta_f^2 \psi\left(\frac{x}{K} - t\right)\right] dt \tag{41}$$

If we restrict ξ to $0 \leq \xi \leq l$, and let $n = 1$ equation (40) gives the pressure distribution on a biconvex airfoil (L being the chord length)

$$\frac{p - p_\infty}{\rho_\infty U_\infty^2} \frac{\beta_f}{\theta_0} = c\left(\frac{x}{K}\right) - \frac{2K}{L} \int_0^{x/K} c\left(\frac{x}{K} - t\right) dt \tag{42}$$

We can compare the pressure distribution from equation (40) with the results from the classical linear theory, which gives the well-known results

Frozen $$\frac{p - p_\infty}{\rho_\infty U_\infty^2} \frac{\beta_f}{\theta_0} = \left(1 - \frac{x}{L}\right) \tag{43a}$$

Equilibrium $$\frac{p - p_\infty}{\rho_\infty U_\infty^2} \frac{\beta_f}{\theta_0} = a^{-1/2}\left(1 - \frac{x}{L}\right) \tag{43b}$$

The expression for pressure from equation (42) has been computed numerically; the result for $a = 2$ is presented in figure 4. When the chord length is very small compared with the relaxation length ($L/K \ll 1$), again the fluid pressure in the flow field of interest ($0 \leq x/L \leq 1$) has no time to adjust itself to the equilibrium value and stays relatively close to the frozen value. As the chord length becomes longer (say $L/K = 1.0$), the fluid pressure begins to take on values closer to equilibrium. For very large chord length, the fluid pressure, after taking on the frozen value at the leading edge, drops rapidly to a variation very close to that of equilibrium flow. At near equilibrium ($L/K \to \infty$) the fluid pressure drops immediately to the equilibrium value as soon as the particle passes the leading edge.

Note also that the fluid particles always start off with the frozen flow properties (at the leading edge) and then tend to acquire the equilibrium values. The pressure actually overshoots the equilibrium value after which it tends toward a variation parallel to the equilibrium curve. As a result the pressure for the nonequilibrium cases has the property that

(a) Free-stream pressure occurs not at the midchord but upstream of it.

(b) The trailing-edge pressure falls between the equilibrium and frozen values.

These two phenomena, the shift of the location of free-stream pressure and the variation of the pressure at the trailing edge, may be useful as means for experimental determination of the relaxation length.

In this last connection, since the optical techniques used in high-temperature gas-flow experiments measure the fluid density rather than pressure, it would be interesting to see how density distribution varies. For a biconvex airfoil, equation (41) gives the density on the wall as

$$\frac{\rho - \rho_\infty}{\rho_\infty}\frac{\beta_f}{\theta_o} = c\left(\frac{x}{K}\right) + \beta_f{}^2\psi\left(\frac{x}{K}\right) - \frac{2K}{L}\int_0^{x/K}\left[c\left(\frac{x}{K} - t\right) + \beta_f{}^2\psi\left(\frac{x}{K} - t\right)\right] dt \qquad (44)$$

This expression has been used to compute the density numerically, and the result is presented in figure 5 for $a = 2$ and $\beta_f = 3$, or $M_f = \sqrt{10} \cong 3$ and $M_e = \sqrt{19} \cong 4$. That essentially the same types of comments on the pressure regarding the frozen, transition, and equilibrium characteristics can be applied here is apparent.

FLOW FIELD AT LARGE ξ

Consider now the flow field at large ξ, which has already been discussed to some extent previously. Since $\xi \to \infty$ corresponds to $s \to 0$, we study the Laplace transforms of the flow quantities at $s \to 0$. Consider, for example, the Laplace transform of $\tilde{v}(\xi,\eta)$, which is, by operating $(\partial/\partial\eta)-(\partial/\partial\xi)$ on $\overline{\varphi}$ from equation (20),

$$\overline{v}(s,\eta) = \overline{\theta}(s)\exp[-s\eta(\sigma - 1)]$$

As s becomes small, $\sigma \to \sqrt{a}$, and $\overline{v}$ approaches

$$\overline{v}(s,\eta) \sim \overline{\theta}(s)\exp[-s\eta(\sqrt{a} - 1)] \tag{45}$$

To interpret this we introduce one more theorem: the shifting theorem, which states

$$\overline{f}(s)\exp(-bs) = L[\tilde{f}(\xi - b)] \tag{46}$$

Applying equation (46) to the right-hand side of equation (45) gives, identifying $\eta(\sqrt{a} - 1)$ as b,

$$L[\tilde{v}(\xi,\eta)] \sim L[\tilde{\theta}(\xi - \sqrt{a}\ \eta + \eta)]$$

$$= L[\tilde{\theta}(x^{+} - \sqrt{a}\ y^{+})]$$

or

$$\tilde{v}(\xi,\eta) \sim \tilde{\theta}(x^{+} - \sqrt{a}\ y^{+})$$

One can verify that $x^{+} - \sqrt{a}\ y^{+}$ = constant are the equilibrium Mach lines. Consequently, at large ξ the velocity $\tilde{v}$ is constant along the equilibrium Mach lines. Another simple computation will verify that, denoting $\alpha = x^{+} - \sqrt{a}\ y^{+}$,

$$\tilde{u}(\xi,\eta) \sim a^{-1/2}\tilde{\theta}(\alpha) \qquad \text{at} \quad \xi \to \infty$$

Therefore at large distances downstream of the initial frozen Mach line the velocities $\tilde{u}$, $\tilde{v}$ (and hence all other flow quantities) are only functions of α. Since at large ξ the flow quantities take essentially the equilibrium values at the wall, we arrive at the expected result: at large ξ the flow quantities take essentially the equilibrium values everywhere and are constant along the equilibrium Mach lines; thus at large ξ the flow approaches the equilibrium limiting case.

The above conclusion is valid when there is no characteristic geometrical length in the problem. When such a length exists, this result must be interpreted with care.

Note that $\xi \equiv (x/K) - (y\beta_f/K)$ can be written as $\xi = l\xi^*$, where ξ^* is defined as $\xi^* \equiv (x - y\beta_f)/L$, (recall $l \equiv L/K$ and L is the characteristic geometrical length).

Thus in the presence of a characteristic geometrical length ξ can approach infinity by either having

(a) $l \to \infty$, ξ^* finite

(b) l finite , $\xi^* \to \infty$

(c) $l \to \infty$,, $\xi^* \to \infty$

There is no question about the validity of cases (a) and (c). Case (b) is valid too, provided the boundary condition is not the recurrent type (such as wavy wall or a wall having an infinite number of wavelets not necessarily of any fixed wavelength). For if the boundary condition is such that new disturbances recur infinitely, then new disturbances can occur in the vicinity of any point no matter how large ξ^* is. In this case the above result is valid only for ξ^* so large (namely, infinity) that it is probably of no practical interest.

FLOW FIELD AT FINITE ξ AND η

We have treated the flow field only at small ξ, small η, and large ξ. The flow field at finite ξ and η has not been discussed because, as mentioned before, the linearized theory is not accurate at large η. Therefore, except for $\xi \to 0$, where the decaying properties of the flow quantities along the initial outgoing frozen Mach line is interesting, and $\xi \to \infty$, where the flow field behaves similar to the equilibrium case, only the flow field for small η was discussed.

Nevertheless, for the sake of completeness, and for those who are interested in working further on this problem (such as obtaining higher approximations), we conclude by showing how the previous results can be extended to η greater than zero.

Note that the Laplace transform of any flow property can be put into the form (eq. (20))

$$\overline{f}(s,\eta) = \overline{f}(s,0)\exp[-s\eta(\sigma - 1]$$

Equation (16c) gives

$$\tilde{f}(\xi,\eta) = \tilde{f}(\xi,0)\lambda(0,\eta) + \int_0^{\xi} \lambda_t(t,\eta)\tilde{f}(\xi - t,0)dt \qquad (47)$$

where

$$\lambda(\xi,\eta) \equiv L^{-1}\left\{s^{-1}\exp[-s\eta(\sigma - 1)]\right\}$$

Berry and Hunter (ref. 7) obtained the inversion for λ for small $(a - 1)$ as

$$\begin{aligned}\lambda(\xi,\eta) = 1 &- (a - 1)\eta \sinh \eta \exp[-(\xi + \eta)][1 + O(a - 1)] \\ &- \frac{2(a - 1)}{\pi}\int_0^{\infty} \frac{z}{(1 + z^2)(1 + az^2)} \sin\left[\frac{a + z^2}{1 + z^2} z(\eta + \xi)\right] \\ &\exp\left[-\frac{a + z^2}{1 + z^2}(\eta + \xi)\right] dz \qquad (48)\end{aligned}$$

Thus equation (47) gives the flow field for all values of ξ and η, with $f(\xi,0)$ already obtained previously, and λ given by equation (48) for small $(a - 1)$. A 5 1 5

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REFERENCES

1. Vincenti, Walter G.: Non-equilibrium Flow Over a Wavy Wall. Jour. Fluid Mech., vol. 6, pt. 4, 1959, pp. 481-496.

2. Clarke, J. F.: The Linearized Flow of a Dissociating Gas. Jour. Fluid Mech., vol. 7, pt. 4, April 1960, pp. 577-595.

3. Moore, Franklin K., and Gibson, Walter E.: Propagation of Weak Disturbances in a Gas Subject to Relaxation Effects. Jour. Aero/Space Sci., vol. 27, no. 2, Feb. 1960, pp. 117-127.

4. Broer, L. J. F.: Characteristics of the Equations of Motion of a Reacting Gas. Jour. Fluid Mech., vol. 4, pt. 3, July 1958, pp. 276-282.

5. Chu, Boa-Teh: Wave Propagation and the Method of Characteristics in Reacting Gas Mixtures with Applications to Hypersonic Flow. WADC TN 57-213, May 1957.

6. Erdelyi, A., ed.: Tables of Integral Transforms. Vol. 1, McGraw-Hill Book Co., Inc., 1954.

7. Berry, D. S., and Hunter, S. C.: The Propagation of Dynamic Stresses in Visco-Elastic Rods. Jour. Mech. and Phys. of Solids, vol. 4, no. 2, Feb. 1956, pp. 72-95. See Corrigenda p. 300.

TABLE I.- FUNCTION $c(\xi;a)$ VERSUS ξ AND a

ξ \ a	1.01	1.1	2	4	6	8	10
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.1	.99952	.99526	.95413	.87182	.80052	.73857	.68458
.2	.99909	.99100	.91559	.77868	.67445	.59409	.53129
.3	.99871	.98718	.88320	.71057	.59330	.51116	.45175
.4	.99835	.98374	.85596	.66044	.54001	.46152	.40743
.5	.99804	.98066	.83304	.62328	.50425	.43049	.38091
.6	.99774	.97789	.81372	.59553	.47972	.41024	.36398
.7	.99748	.97543	.79745	.57465	.46252	.39648	.35256
.8	.99725	.97317	.78372	.55882	.45018	.38679	.34451
.9	.99704	.97116	.77213	.54671	.44115	.37975	.33862
1.0	.99685	.96936	.76235	.53738	.43442	.37449	.33420
1.2	.99652	.96629	.74707	.52446	.42535	.36738	.32817
1.4	.99625	.96381	.73612	.52643	.41981	.36298	.32440
1.6	.99603	.96181	.72825	.51128	.41627	.36013	.32194
1.8	.99585	.96020	.72257	.50789	.41392	.35822	.32029
2.0	.99570	.95890	.71845	.50561	.41232	.35692	.31916
2.2	.99558	.95784	.71546	.50404	.41120	.35600	.31887
2.4	.99548	.95700	.71328	.50295	.41042	.35535	.31780
2.6	.99540	.95632	.71169	.50217	.40985	.35489	.31739
2.8	.99533	.95577	.71052	.50161	.40944	.35455	.31709
3.0	.99528	.95532	.70966	.50120	.40914	.35430	.31688
3.5	.99518	.95455	.70835	.50059	.40869	.35392	.31655
4.0	.99513	.95410	.70773	.50030	.40847	.35374	.31639
4.5	.99509	.95383	.70742	.50016	.40837	.35365	.31631
5.0	.99507	.95368	.70727	.50008	.40831	.35361	.31627
5.5	.99506	.95359	.70719	.50004	.40828	.35358	.31625
6.0	.99505	.95354	.70715	.50002	.40826	.36357	.31624
∞	.99504	.95346	.70711	.50000	.40825	.35355	.31623

TABLE II.- FUNCTION $\psi(\xi;a)$ VERSUS ξ AND a

ξ \ a	1.01	1.1	2	4	6	8	10
0	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000	1.00000
.1	1.00048	1.00475	1.04701	1.13773	1.22441	1.30740	1.38703
.2	1.00091	1.00904	1.08851	1.25396	1.40617	1.54728	1.67902
.3	1.00130	1.01291	1.12519	1.35255	1.55529	1.73886	1.90718
.4	1.00165	1.01641	1.15761	1.43655	1.67900	1.89474	2.09024
.5	1.00197	1.01956	1.18629	1.50842	1.78260	2.02349	2.24009
.6	1.00225	1.02241	1.21168	1.57016	1.87006	2.13110	2.36460
.7	1.00251	1.02498	1.23416	1.62338	1.94441	2.22192	2.46926
.8	1.00275	1.02730	1.25408	1.66940	2.00799	2.29916	2.55803
.9	1.00296	1.02939	1.27174	1.70931	2.06262	2.36528	2.63387
1.0	1.00316	1.03128	1.28740	1.74402	2.10979	2.42218	2.69904
1.2	1.00348	1.03453	1.31363	1.80069	2.18616	2.51403	2.80406
1.4	1.00376	1.03717	1.33432	1.84410	2.24417	2.58358	2.88347
1.6	1.00398	1.03932	1.35067	1.87758	2.28865	2.63679	2.94416
1.8	1.00417	1.04108	1.36361	1.90355	2.32301	2.67783	2.99093
2.0	1.00432	1.04251	1.37387	1.92380	2.34971	2.70969	3.02722
2.2	1.00444	1.04368	1.38201	1.93964	2.37056	2.73454	3.05551
2.4	1.00454	1.04463	1.38849	1.95209	2.38690	2.75401	3.07767
2.6	1.00462	1.04540	1.39364	1.96191	2.39976	2.76932	3.09509
2.8	1.00469	1.04603	1.39774	1.96966	2.40991	2.78140	3.10882
3.0	1.00474	1.04654	1.40102	1.97579	2.41794	2.79094	3.11968
3.5	1.00483	1.04745	1.40661	1.98617	2.43148	2.80704	3.13799
4.0	1.00490	1.04799	1.40981	1.99204	2.43914	2.81614	3.14833
4.5	1.00493	1.04832	1.41165	1.99540	2.44351	2.82133	3.15422
5.0	1.00495	1.04852	1.41272	1.99733	2.44602	2.82431	3.15760
5.5	1.00497	1.04863	1.41334	1.99844	2.44747	2.82603	3.15956
6.0	1.00498	1.04870	1.41370	1.99909	2.44831	2.82702	3.16069
∞	1.00499	1.04881	1.41421	2.00000	2.44949	2.82842	3.16228

A
5
1
5

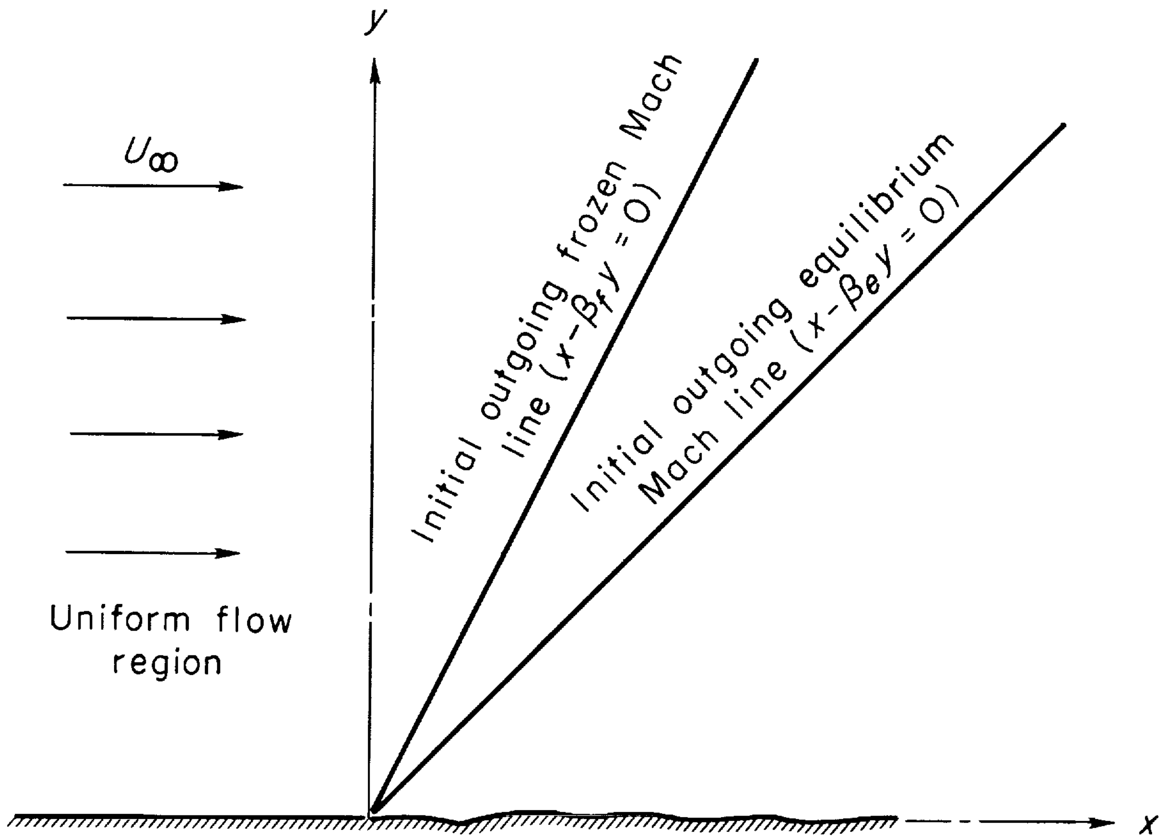

Figure 1.- Linearized supersonic flow past an arbitrary boundary.

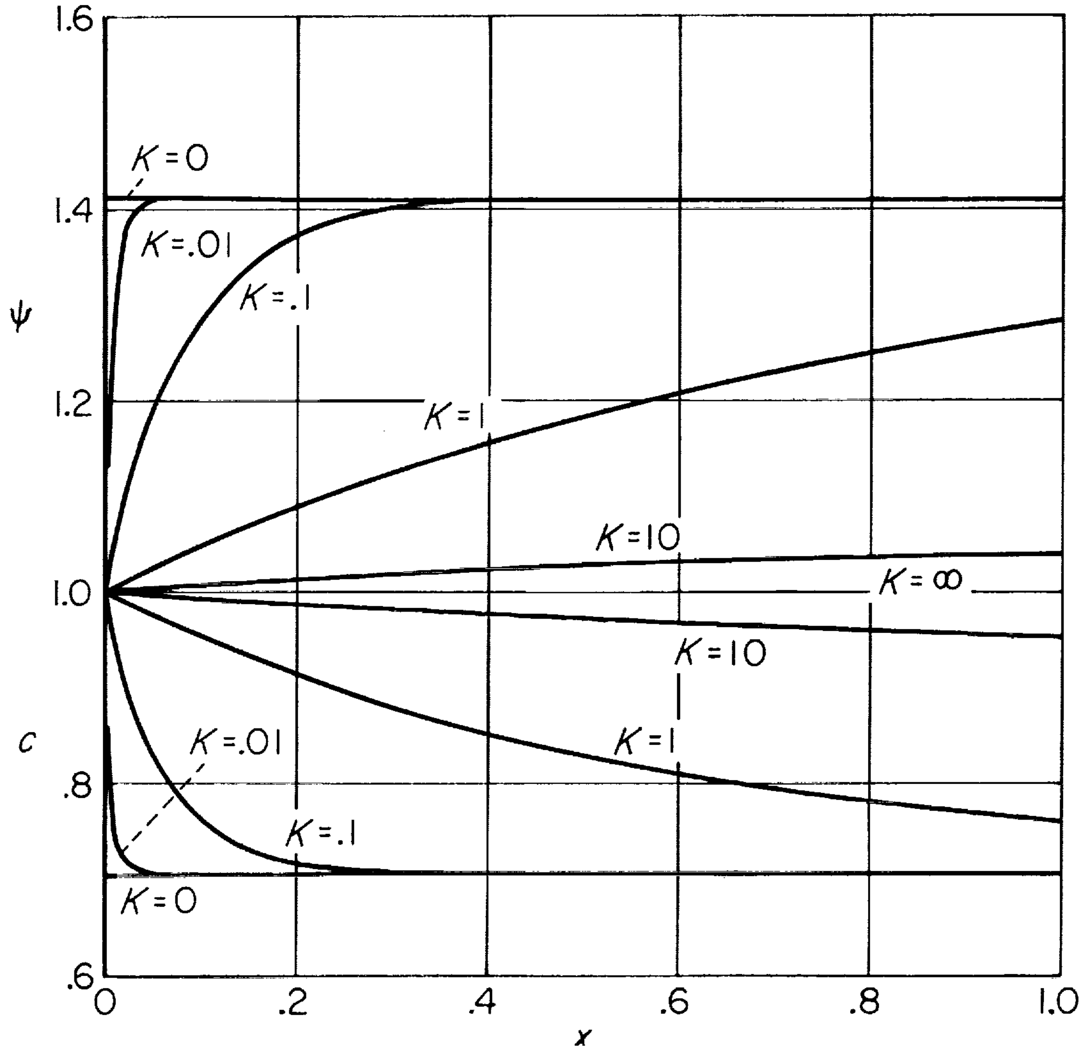

Figure 2.- The functions c and ψ versus x for several values of K; $(M_e^2 - 1)/(M_f^2 - 1) = 2$.

A
5
1
5

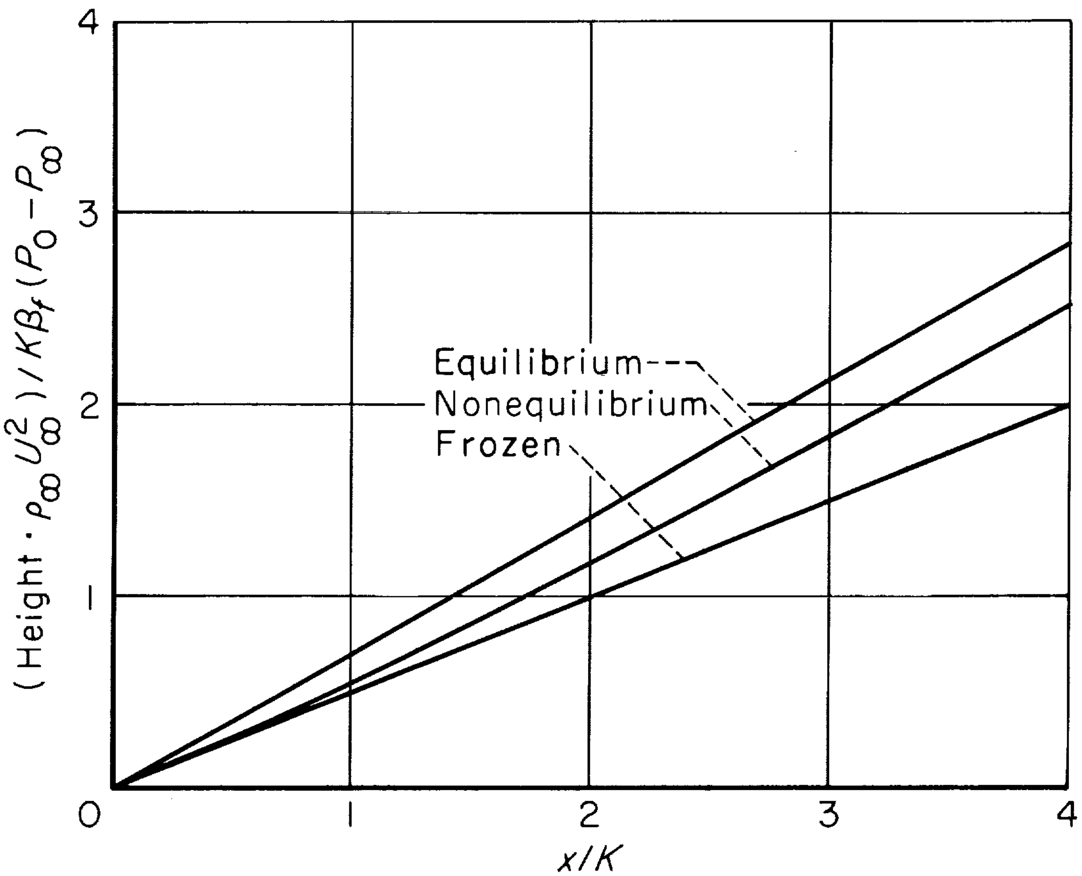

Figure 3.- Height of a boundary when the pressure on the boundary is constant, p_o; $(M_e^2 - 1)/(M_f^2 - 1) = 2$.

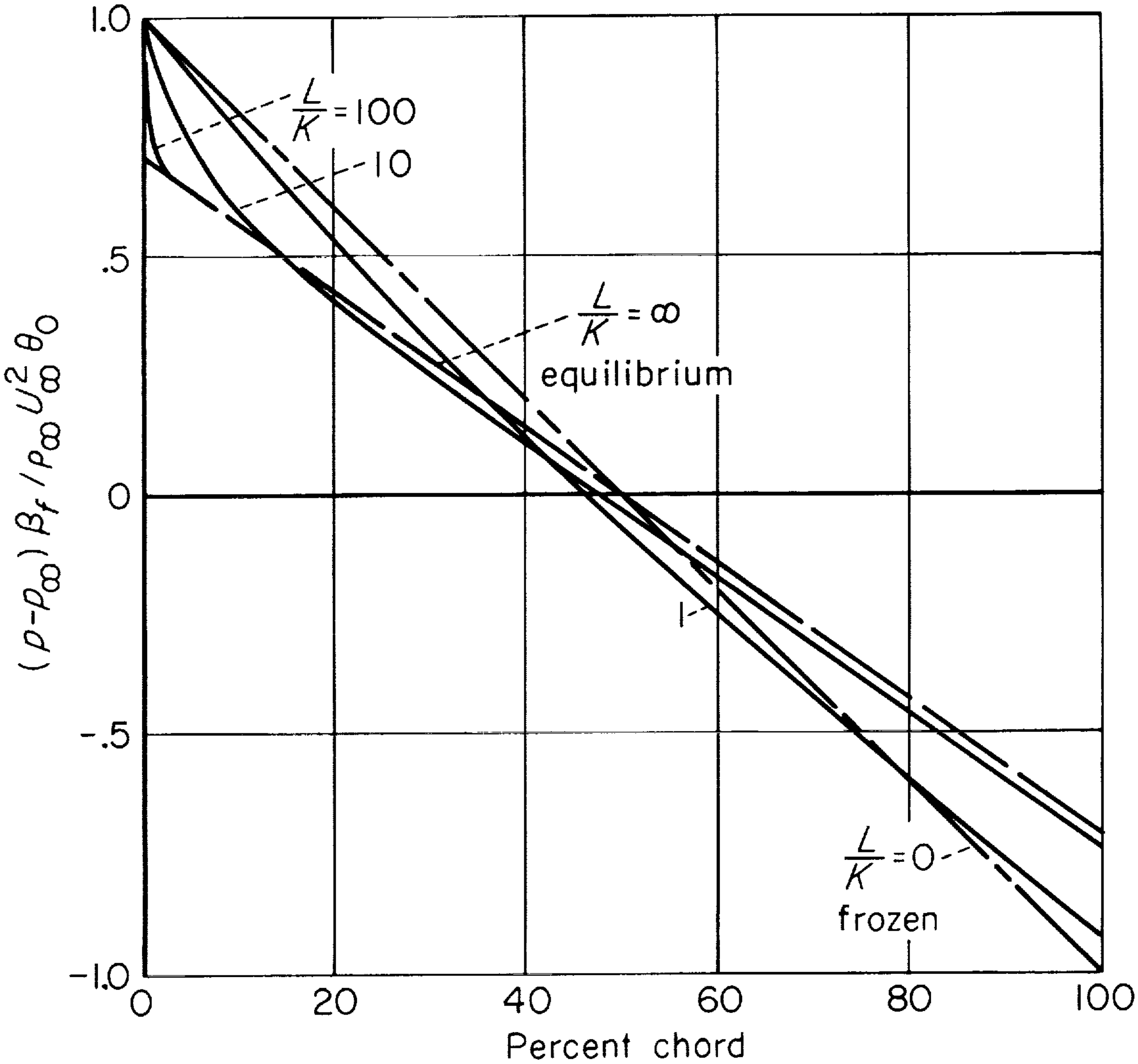

Figure 4.- Pressure distribution on a biconvex airfoil versus several values of L/K; $(M_e^2 - 1)/(M_f^2 - 1) = 2$.

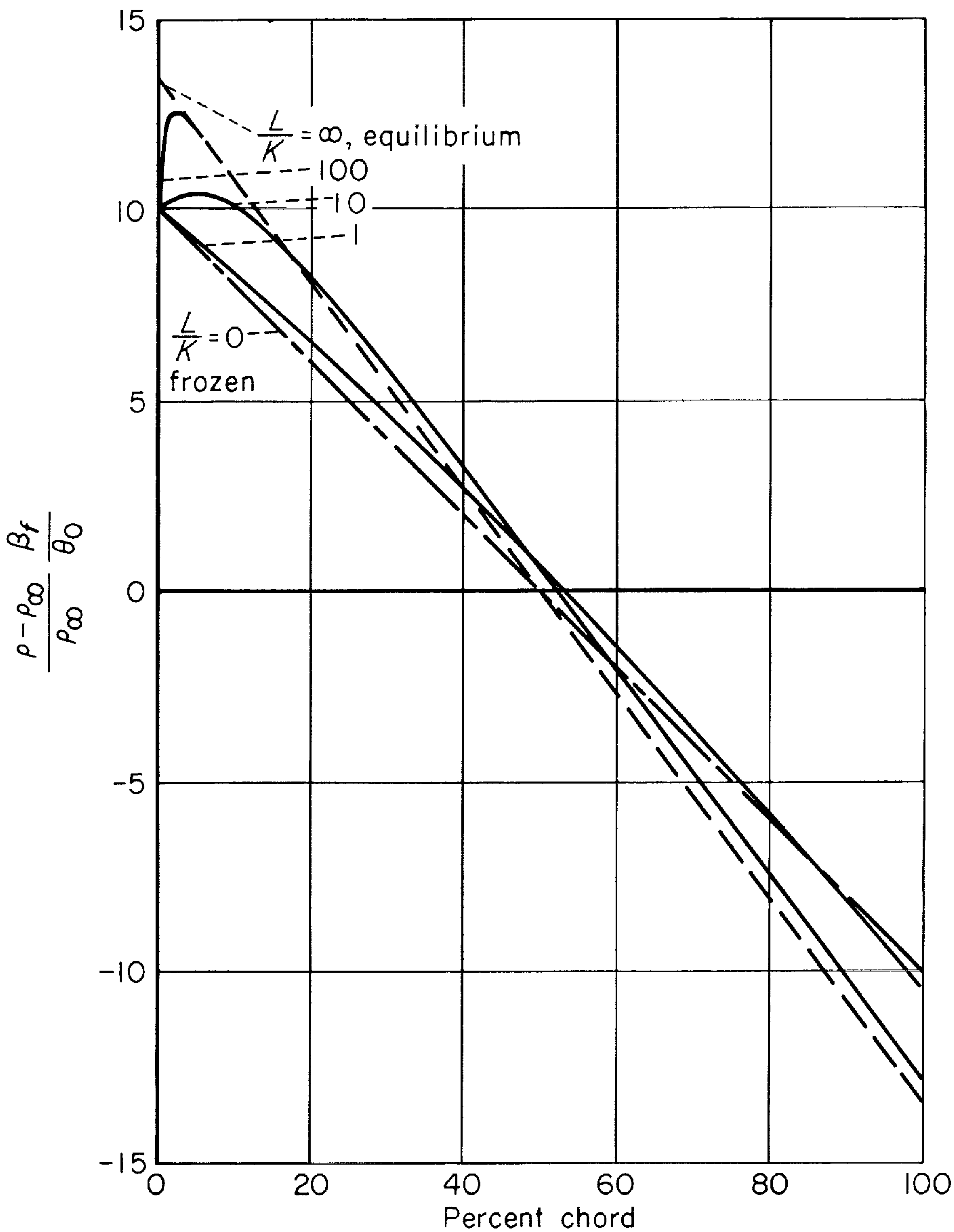

Figure 5.- Density distribution on a biconvex airfoil versus several values of L/K; $(M_e^2 - 1)/(M_f^2 - 1) = 2$ and $M_f = \sqrt{10}$.